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Nilpotency of square matrices with non-negative elements

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Abstract We give two equivalent decision procedures for nilpotency of square matrices with non-negative elements, which do not involve computation of the matrix eigenvalues. To the best of our knowledge these procedures have not been formalised elsewhere: we have decided to produce this brief technical report as a future reference.

A matrix is said to be *nilpotent of order* k if k is the least natural number such that: *Nilpotent matrix*

$$M^k = 0$$

We aim at a decision procedure to enable us to decide by visual inspection if a matrix $M \in \mathbb{R}^{n \times n}$ is nilpotent. In particular we are interested in matrices characterised by non-negative elements, and in this case we are sure that nilpotency can be inferred from considerations on the positions of null elements.

The most trivial observations are the following:

- if there is not at least one null column in M, then it is not nilpotent; *Null column*
- if all columns of M are null (*i.e.* M is the null matrix), then it is nilpotent, and its nilpotency order is 1. *Null matrix*

In the above cases M is said to be *terminal*, as its nilpotency is trivially decidable. *Terminal matrix*

We are now going to describe informally how to understand where the other null elements should be so that a non-terminal matrix M is nilpotent.
If $\mathrm{Nilp}(M)$ denotes the nilpotency order of M, we have that:

$$\forall \underline{v} \in \mathbb{R}^n \bullet M^{\mathrm{Nilp}(M)}\, \underline{v} = \underline{0}$$

If we imagine that $\underline{v} = (\mu_1, \mu_2, \ldots, \mu_n)^T$ represents a distribution of mass in n different places, then M can describe some event that leads to a new mass distribution, that depends on the previous one. The new distribution $\underline{v}' = (\mu'_1, \mu'_2, \ldots, \mu'_n)^T$ is such that: *Informal description*

$$\mu'_i = m_{i1}\mu_1 + m_{i2}\mu_2 + \cdots + m_{in}\mu_n$$

and hence we can see this in the following way:

- if $m_{ij} = 1$, a copy of the mass in the j-th place is moved to the new i-th place: $\mu'_i = \mu_j$;
- if $m_{ij} = 0$, a copy of the mass in the j-th place is trashed into the bin: $\mu'_i = 0$;
- if $m_{ij} < 1$, a copy of the mass in the j-th place is shrunk and moved to the new i-th place and part is trashed into the bin: $\mu'_i = m_{ij}\mu_j < \mu_j$;

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- if $m_{ij} > 1$, a copy of the mass in the j-th place is enlarged and moved to the new i-th place: $\mu'_i = m_{ij}\mu_j > \mu_j$.

The subsequent application of the matrix M causes the mass to "follow" a certain path from its initial position to some other position: if every path leads to the bin, then we are happy to say that M is nilpotent.
We can describe the path in quite fine detail, as:

- if some k-th column is null, then all copies of the mass originally in the k-th place are thrown away after the first application of M;
- if some h-th column is null with the exception of the element m_{kh} (k as above), then all copies of the mass originally in the h-th place are thrown away in two steps;
- if some g-th column is null with the exception of some elements among m_{hg} and m_{kg} (h,k as above), then all copies of the mass originally in the g-th are thrown away in at most three steps;
- ...

In the next section we formalise this procedure, we then provide a different procedure (with a proof), and finally provide an equivalence proof of the two procedures.

1 First procedure

Set T_i

Let M be non-terminal; given a succession $Z_1, Z_2, \ldots$ of sets let T_i be the set given by the union of the first i sets in the succession:

$$T_i \triangleq \bigcup_{1 \le j \le i} Z_j$$

Procedure 1

Let $Z_1, Z_2, \ldots, Z_{(k+1)}$ be the succession such that:

- Z_1 is the set of the indices of all null columns of M;
- if $Z_i \neq \varnothing$, then Z_{i+1} is the set of indices identifying the non-null columns of M such that the only non-null elements are those with row index contained in T_i or in a subset thereof;
- $Z_{(k+1)} = \varnothing$.

Then M is nilpotent if and only if $T_k = \{1, 2, \ldots, n\}$; moreover in this case it is $\text{Nilp}(M) = k$.
If we look at the succession $Z_1, Z_2, \ldots, Z_{(k+1)}$ from the perspective used in the informal description, we have that:

- Z_1 indicates all places from where the mass is thrown away;
- more in general, Z_i indicates what are the places from where the mass is thrown away in i steps;
- therefore T_i indicates what are the places from where the mass is thrown away in at most i steps.

2 Second procedure

Reduced matrix

Let M be non-terminal; M^R is the *reduction* (or *reduced matrix*) of M, if it can be obtained from M by removing all rows and columns with indices contained in Z_1, where Z_1 is the set of all indices of null columns (as in procedure 1).

Succession

Let $M_1, M_2, \ldots, M_k$ be the succession such that:

- $M_1 = M$;
- if M_i is not terminal, then $M_{i+1} = M_i^R$;

- M_k is terminal.

Then M is nilpotent if and only if M_k is nilpotent; moreover in this case all the matrices in the succession are nilpotent, and their nilpotency orders verify the relations

$$\text{Nilp}(M_k) = 1 \qquad \text{Nilp}(M_{(i-1)}) = \text{Nilp}(M_i) + 1$$

and therefore

$$\text{Nilp}(M) = k$$

Procedure 2

The procedure to decide on the nilpotency of M is therefore to start with the matrix M and subsequently cross out all columns containing exclusively null elements along with the rows with the same index: by iterating the process we eventually end up with a reduction M_k of M which contains only non-null columns or only null columns, and therefore the nilpotency of M_k is trivially decidable.

3 Proofs

3.1 Proof of procedure 2

Let P be a permutation matrix such that:

$$PMP^{-1} = \begin{pmatrix} 0_{r\times r} & H_{r\times s} \\ 0_{s\times r} & M^R \end{pmatrix}$$

which contains r null columns, with indices less or equal than r, and $s = n - r$ non-null columns, with indices greater than r.

We then have that:

$$PM^hP^{-1} = (PMP^{-1})^h = \begin{pmatrix} 0_{r\times r} & H_{r\times s}(M^R)^{(h-1)} \\ 0_{s\times r} & M^R(M^R)^{(h-1)} \end{pmatrix}$$

We also have that $M^h = 0 \Leftrightarrow PM^hP^{-1} = 0$, and therefore:

$$M^h = 0 \Leftrightarrow \begin{pmatrix} H_{r\times s} \\ M^R \end{pmatrix} (M^R)^{(h-1)} = 0$$

which can be true if and only if $(M^R)^{(h-1)} = 0$, given that in the matrix

$$\begin{pmatrix} H_{r\times s} \\ M^R \end{pmatrix}$$

all columns are non-null, and all elements are non-negative.

We can therefore conclude that:

- M is nilpotent if and only if M^R is nilpotent;
- $\text{Nilp}(M) = \text{Nilp}(M^R) + 1$.

It should be noted that removing the assumption $m_{ij} \geq 0$ allows us to conclude that in general

$$\text{Nilp}(M^R) \leq \text{Nilp}(M) \leq \text{Nilp}(M^R) + 1$$

Observation Given that $M^h \to 0 \Leftrightarrow PM^hP^{-1} \to 0$, we can show with a similar proof that

$$M^h \to 0 \Leftrightarrow (M^R)^h \to 0$$

3.2 Proof of equivalence of the procedures

The set Z_i of procedure 1, contains the indices of the columns and rows which are crossed out at the $(i+1)$-th step of procedure 2; hence the equivalence ot the procedures is proved by the following remarks:

- for $i = 2, \ldots, k$ the matrix M_i of procedure 2 can be obtained from M by removing all rows and columns with indices contained in T_{i-1} of procedure 1;
- the matrix M_k is terminal and null if and only if $Z_k \neq \varnothing$ and $T_k = \{1, \ldots, n\}$; moreover in this case it is $Z_{k+1} = \varnothing$;
- the matrix M_k is terminal and non null if and only if $Z_k = \varnothing$ and $T_{(k-1)} \neq \{1, \ldots, n\}$.